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The Dirichlet-Multinomial Model for Multivariate Randomized Response Data and Small Samples

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In survey sampling the randomized response (RR) technique can be used to obtain truthful answers to sensitive questions. Although the individual answers are masked due to the RR technique, individual (sensitive) response rates can be estimated when observing multivariate response data. The beta-binomial model for binary RR data will be generalized to handle multivariate categorical RR data. The Dirichlet-multinomial model for categorical RR data is extended with a linear transformation of the masked individual categorical-response rates to correct for the RR design and to retrieve the sensitive categorical-response rates even for small data samples. This specification of the Dirichlet-multinomial model enables a straightforward empirical Bayes estimation of the model parameters. A constrained-Dirichlet prior will be introduced to identify homogeneity restrictions in response rates across persons and/or categories. The performance of the full Bayes parameter estimation method is verified using simulated data. The proposed model will be applied to the college alcohol problem scale study, where students were interviewed directly or interviewed via the randomized response technique about negative consequences from drinking.

The data collection through surveys based on direct-questioning methods has been the most common way. The direct-questioning techniques are usually assumed to provide the necessary level of reliability when measuring opinions, attitudes, and behaviors. However, individuals with different types of response behavior who are confronted with items about sensitive issues of human life regarding ethical (stigmatizing) and legal (prosecution) implications are reluctant to supply truthful answers.

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Tourangeau, Rips, and Rasinski (2000), and Tourangeau and Yan (2007) argued that socially desirable answers and refusals are to be expected when asking sensitive questions directly.

Warner (1965), and Greenberg, Abu-Ela, Simmons, and Horvitz (1969) developed RR techniques to obtain truthful answers to sensitive questions in such a way that the individual answers are protected but population characteristics can be estimated. These techniques are based on univariate RR data. Recently, RR models have been developed to analyze multivariate response data, where the item responses are nested within the individual. Although the individual answers are masked due to the RR technique, individual (sensitive) characteristics can be estimated when observing multivariate RR data. Fox (2005) and Böckenholt and van der Heijden (2007) introduced item response models for binary RR data. The applications are focusing on surveys where the items measure an underlying sensitive construct. The so-called randomized item response models have been extended to handle categorical RR data by Fox and Wyrick (2008) and De Jong, Pieters, and Fox (2010). The class of randomized item response models are meant for large-scale survey data, since person as well as item parameters need to be estimated (Fox and Wyrick, 2008). For categorical item response data, more than 500 respondents are often needed to obtain stable parameter estimates. Furthermore, the randomized item response data are less informative than the direct-questioning data, since the RR technique engenders additional random noise to the data. Fox (2008) proposed a beta-binomial model for analyzing multivariate binary RR data, which enables the computation of individual response estimates without requiring a large-scale data set. The beta-binomial model has several advantages like a simple interpretation of the model parameters, stable parameter estimates for relatively small data sets, and a straightforward empirical Bayes estimation method.

Here, a Dirichlet-multinomial model is proposed for handling multivariate categorical RR data such that individual category-response rates can be estimated. The individual observed RR data consist of a number of randomized responses per category. Each individual set of observed numbers are assumed to be multinomially distributed given the individual category-response rates. The individual category-response rates are assumed to follow a Dirichlet distribution. The individual response rates are related to the observed randomized responses, which make them not useful for the inferences basing on regular statistical approaches. However, it will be shown that the individual category-response rates are linearly related to the model-based (true) category-response rates. The latter one relates to the latent responses, which are expected under the model when the responses are not masked due to the randomized response technique. The parameters of the linear transformation are design parameters and are known characteristics of the randomizing device that is used to mask the individual answers. The transformed categorical-response rates will provide information about the latent individual characteristic that is measured by the survey items. Analytical expressions of the posterior mean and standard deviation of the true individual categorical-response rates will be given. The expressions can be used for estimation given prior knowledge or empirical Bayes estimates of the population response rates. Furthermore, a WinBUGS implementation is given for a full Bayes estimation of the model parameters.

To model and to identify constraints of homogeneity in category-response rates, the restricted-Dirichlet prior (Schafer, 1997) is used. The restriction on the Dirichlet prior can be used to identify effects of the randomized response mechanism across individuals, groups of individuals, and response categories.

In the next section, the randomized response technique is described in a multiple-item setting. The beta-binomial model is described for multivariate binary RR outcomes. Then, as a generalization, the Dirichlet-multinomial model is presented for multivariate categorical RR data. Properties of the conditional posterior distribution of the true individual categorical-response rates are derived given observed randomized response data. Then, empirical and full Bayes methods are proposed to estimate all model parameters. A simulation study is given, where the properties of the estimation methods are examined. Finally, the model will be used to analyze data from a college alcohol problem scale survey, where U.S. college students were asked about their alcohol drinking behavior with and without using the randomized response technique. The restricted-Dirichlet prior will be used to test assumptions of homogeneity over persons and response categories. In particular, it will be shown that the effect of the RR method varies over response categories, where the RR effect will be the highest for the most sensitive response option.

MULTIVARIATE RANDOMIZED RESPONSE TECHNIQUES

In Warner's RR technique (Warner, 1965) for univariate binary response data, in the data collection procedure a randomizing device (RD) is introduced. For each respondent the RD directs the choice of one of two logically opposite questions. This sampling design guarantees the confidentiality of the individual answers, since they cannot be related directly to one of the opposite questions.

Greenberg et al. (1969) proposed the unrelated question technique, where the outcome of the RD refers to the study-related sensitive question or an irrelevant unrelated question. The RD is specified in such a way that the sensitive question is selected with probability ϕ_1 and the unrelated question with probability $1-\phi_1$. This RR method is extended to a forced response method (Edgell, Himmelfarb, and Duchan, 1982), where the unrelated question is not specified but an additional RD is used to generate a forced answer. Each observed individual answer is protected, since it cannot be revealed whether it is a true answer to the sensitive question or a forced answer generated by the RD. As a result, the observed RR answers are polluted by forced responses.

Let $RD = 1$ denote the event that an answer to the sensitive question is required and $P(RD = 1) = \phi_{1k}$ and $RD = 0$ otherwise. A forced positive response to item k is generated with probability ϕ_{2k}. For a multiple-item survey, the probability of a positive RR of respondent i, given a forced response sampling design, can be stated as

$$P\left(Y_{ik} = 1 \mid \phi, p_{ik}\right) = P(RD = 1)p_{ik} + (1 - P(RD = 1))\phi_{2k}, \qquad (1)$$

where the true response rate of person i to item k is denoted as p_{ik}. Note that the response model for the RR data is a two-component mixture model. For the first component the sensitive question needs to be answered and for the second component a forced response needs to be generated. Thus, the randomized response probability equals the true or the forced response probability depending on the RD outcome. With $\phi_{1k} > 1/2$, for all k, the data contain sufficient information to make inferences about the true response rates.

The multiple items will be assumed to measure an underlying individual response rate (e.g., alcohol dependence, academic fraud) such that $p_{ik} = p_i$ for all k. This individual response rate can be estimated from the multivariate RR data. Note that in a multivariate setting the RD characteristics are allowed to vary over items such that the proportion of forced responses can vary over items. In practice, the sensitivity of the items may vary although they relate to the same sensitive latent characteristic. This variation in sensitivity can be controlled by adjusting the RD characteristics, which are under the control of the interviewer.

The forced response model in Equation (1) can be extended to handle polytomous multivariate RR data. Let $\phi_{2k}(c)$ denote the probability of a forced response in category c for $c = 1, \ldots, C_k$ such that the number of response categories may vary over items. The categorical-response rates of individual i are denoted as $p_i(1), \ldots, p_i(C_k)$, which represent the probabilities of honest (true) responses corresponding to the response categories of item k. The probability of an observed randomized response of individual i in category c of item k can be stated as,

$$P(Y_{ik} = c \mid \phi, p_{ik}) = \phi_{1k} p_i(c) + (1 - \phi_{1k}) \phi_{2k}(c). \qquad (2)$$

This forced RR model for categorical data can be used to measure individual categorical response rates related to a sensitive characteristic. The individual answers are not known but the multivariate data make it possible to retrieve information about latent individual characteristics.

THE BETA-BINOMIAL MODEL FOR MULTIVARIATE BINARY RR DATA

Let each participant $i = 1, \ldots, N$ respond to $k = 1, \ldots, K$ binary items. The observations $u_{i1}, \ldots, u_{iK}$ represent the answers of the i^{th} participant to the K items. The response observations are assumed to be Bernoulli distributed given response rate p_i for individual i. The observations are assumed to be independently distributed given the response rate. Therefore, the sum of individual response observations is binomially distributed with parameters K and p_i.

It is to be expected that the response rates vary over participants. This variation is modeled by means of a beta distribution with parameters $\tilde{\alpha}$ and $\tilde{\beta}$, which specify the distribution of the response rates. This leads to the following hierarchical model for the multivariate binary response observations,

$$\begin{aligned} U_{i.} \mid p_i &\sim \mathcal{BIN}(K, p_i), \\ p_i \mid \tilde{\alpha}, \tilde{\beta} &\sim \mathcal{B}(\tilde{\alpha}, \tilde{\beta}), \end{aligned}$$

where $U_{i.} = \sum_k U_{ik}$.

Within a Bayesian modeling approach, the beta prior distribution for parameter p_i is a conjugated prior when the data are binomially distributed given the response rate. In that case, the posterior distribution of the response rate is also a beta distribution. That is,

$$\begin{aligned} p\left(p_i \mid u_{i\cdot}, \tilde{\alpha}, \tilde{\beta}\right) &= \frac{f(u_{i\cdot} \mid p_i)\pi(p_i \mid \tilde{\alpha}, \tilde{\beta})}{\int f(u_{i\cdot} \mid p_i)\pi(p_i \mid \tilde{\alpha}, \tilde{\beta})dp_i} \\ &= \frac{\Gamma(K + \tilde{\alpha} + \tilde{\beta})}{\Gamma(u_{i\cdot} + \tilde{\alpha})\Gamma(K - u_{i\cdot} + \tilde{\beta})} p_i^{u_{i\cdot} + \tilde{\alpha} - 1}(1 - p_i)^{K - u_{i\cdot} + \tilde{\beta} - 1}, \end{aligned}$$

which can be recognized as a beta density with parameters $u_{i\cdot} + \tilde{\alpha}$ and $K - u_{i\cdot} + \tilde{\beta}$. The posterior mean and the variance are

$$E(p_i \mid u_{i.}, \tilde{\alpha}, \tilde{\beta}) = \frac{u_{i.} + \tilde{\alpha}}{K + \tilde{\alpha} + \tilde{\beta}},$$

$$Var(p_i \mid u_{i.}, \tilde{\alpha}, \tilde{\beta}) = \frac{(u_{i.} + \tilde{\alpha})(K - u_{i.} + \tilde{\beta})}{(K + \tilde{\alpha} + \tilde{\beta} + 1)(K + \tilde{\alpha} + \tilde{\beta})^2},$$

respectively. It follows that posterior inferences can be directly made when knowing the population parameters $\tilde{\alpha}$ and $\tilde{\beta}$.

In a forced response design, the observations $\mathbf{u}$ are masked and randomized responses $\mathbf{y}$ are observed. The RD specifies the probabilities governing this randomization process such that an honest response is to be given with probability ϕ_1 and a positive forced response with probability $(1 - \phi_1)\phi_2$. The probability of observing a positive response from participant i to item k is related to the true response by the following expression:

$$\begin{aligned} P(Y_{ik} = 1 \mid p_i) &= \phi_1 f(u_{ik} \mid p_i) + (1 - \phi_1)\phi_2 \\ &= \phi_1 p_i + (1 - \phi_1)\phi_2 = \Delta(p_i). \end{aligned}$$

It can be seen that the forced response design corresponds with a linear transformation of the response rate. This linear transformation function, $\Delta(.)$, operates on the individual response rate of the true responses. Therefore, the beta-binomial model accommodates the forced response sampling mechanism by modeling the linearly transformed response rates; that is,

$$\begin{aligned} Y_{i.} \mid p_i &\sim \mathcal{BIN}\left(K, \Delta(p_i)\right), \\ \Delta(p_i) &\sim \mathcal{B}(\alpha, \beta), \end{aligned}$$

where the transformation parameters ϕ_1 and ϕ_2 are characteristics of the RD and are known a priori.

A population distribution is specified for the transformed response rates. The transformed response rates are a priori beta distributed, which is the conjugated prior for the binomially distributed likelihood. As a result, the posterior distribution of the transformed response rates is beta distributed with parameters $y_{i\cdot} + \alpha$ and $K - y_{i\cdot} + \beta$.

The posterior expected response rate given the randomized responses can be expressed as

$$\begin{aligned} E\left(\Delta(p_i) \mid y_{i.}, \alpha, \beta\right) = \frac{y_{i.} + \alpha}{K + \alpha + \beta} &= \Delta\left(E(p_i) \mid y_{i.}, \alpha, \beta\right) \\ &= \phi_1 E\left(p_i \mid y_{i.}, \alpha, \beta\right) + (1 - \phi_1)\phi_2, \end{aligned}$$

using that the expected value of the linearly transformed response rate equals the linearly transformed expected response rate. As a result, the posterior expected value of the (true) response rate can be expressed as

$$E\left(p_i \mid y_{i.}, \alpha, \beta\right) = \phi_1^{-1}\left(\frac{y_{i.} + \alpha}{K + \alpha + \beta}\right) + (1 - \phi_1^{-1})\phi_2. \tag{3}$$

In the same way, an expression can be found for the posterior variance of the true response rate,

$$Var\left(p_i \mid y_{i.}, \alpha, \beta\right) = \frac{(y_{i.} + \alpha)(K - y_{i.} + \beta)}{\phi_1^2 (K + \alpha + \beta + 1)(K + \alpha + \beta)^2}.$$

There are two straightforward methods for estimating the hyperparameters α and β, the method of moments and the method of maximizing the marginal likelihood. Given the estimated hyperparameters, empirical

Bayes estimates of the response rates can be derived by inserting the hyperparameter estimates into Equation (3). Furthermore, the estimation of confidence intervals and Bayes factors is described in Fox (2008).

THE DIRICHLET-MULTINOMIAL MODEL FOR MULTIVARIATE CATEGORICAL RR DATA

The number of responses per response category over items for person i are stored in a vector $\mathbf{u}_{i.} = (u_{i.1}, \ldots, u_{i.C})^t$, where $u_{i.c} = \sum_k u_{ikc}$ for $c = 1, \ldots, C$. They represent the number of choices per response category over items. In the college alcohol study we will present in Section 7.2, the data represent the frequency of alcohol-related negative consequences. In marketing research, Goodhardt, Ehrenberg, and Chatfield (1984) considered data about individual number of purchases per brand in a time period. In social research, Wilson and Chen (2007) considered frequencies to television viewing questions from the High School and Beyond survey study in the United States. Their item-based test is assumed to measure the daily television viewing habit and interest is focused on time-specific population response rates.

The number of responses per category given the category response rates are assumed to be independently distributed. They can be modeled by a multinomial distribution with parameters K and category response rates $p_{i1}, \ldots, p_{iC}$. For respondent i, the contribution to the likelihood is

$$f(\mathbf{u}_{i.} \mid \mathbf{p}_i) = \frac{K!}{\prod_c u_{i.c}!} \prod_c p_{ic}^{u_{i.c}}.$$

The variability in the vectors of response counts is often higher than can be accommodated by the multinomial distribution. Therefore, individual variation in category response rates is modeled by a Dirichlet distribution with parameters $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_C)$, which is represented by

$$\pi(\mathbf{p}_i \mid \tilde{\alpha}) = \frac{\Gamma(\tilde{\alpha}_0)}{\prod_c \Gamma(\tilde{\alpha}_c)} \prod_c p_{ic}^{\tilde{\alpha}_c - 1}.$$

where $\tilde{\alpha}_0 = \sum_c \tilde{\alpha}_c$. The within-individual and between-individual variability in response rates is described by a Dirichlet-multinomial model; that is,

$$\begin{aligned} U_{i.1}, \ldots, U_{i.C} \mid p_{i1}, \ldots, p_{iC} &\sim Mult(K, p_{i1}, \ldots, p_{iC}), \\ p_{i1}, \ldots, p_{iC} &\sim \mathcal{D}(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_C), \end{aligned}$$

where $U_{i.c} = \sum_k U_{ikc}$ for $c = 1, \ldots, C$. The compact form of this expression can be written in terms of vector notation

$$\begin{aligned} \mathbf{U}_{i.} \mid \mathbf{p}_i &\sim Mult(K, \mathbf{p}_i), \\ \mathbf{p}_i &\sim \mathcal{D}(\tilde{\alpha}), \end{aligned}$$

where $\mathbf{U}_{i.} = (U_{i.1}, \ldots, U_{i.C})^t$.

The Dirichlet distribution is a conjugate prior for the parameters of the multinomially distributed responses. Therefore, the conditional posterior distribution of the category response rates is a Dirichlet distribution, which is represented by

$$\begin{aligned} p(\mathbf{p}_i \mid \mathbf{u}_{i.}, \tilde{\alpha}) &= \frac{f(\mathbf{u}_{i.} \mid \mathbf{p}_i)\pi(\mathbf{p}_i \mid \tilde{\alpha})}{\int f(\mathbf{u}_{i.} \mid \mathbf{p}_i)\pi(\mathbf{p}_i \mid \tilde{\alpha})d\mathbf{p}_i} \\ &= \frac{\Gamma(K + \tilde{\alpha}_0)}{\prod_c \Gamma(u_{i.c} + \tilde{\alpha}_c)} \prod_c p_{ic}^{u_{i.c} + \tilde{\alpha}_c - 1}. \end{aligned}$$

The posterior mean and the variance of the category response rates of individual i equals

$$E(p_{ic} \mid \mathbf{u}_{i.}, \tilde{\alpha}) = \frac{u_{i.c} + \tilde{\alpha}_c}{K + \tilde{\alpha}_0}$$

and

$$Var(p_{ic} \mid \mathbf{u}_{i.}, \tilde{\alpha}) = \frac{(u_{i.c} + \tilde{\alpha}_c)(K + \tilde{\alpha}_0 - (u_{i.c} + \tilde{\alpha}_c))}{(K + \tilde{\alpha}_0 + 1)(K + \tilde{\alpha}_0)^2},$$

respectively, where the prior parameters $\tilde{\alpha}$ are unknown.

According to Equation (2), the probability of an observed randomized response in category c for item k can be expressed as

$$\begin{aligned} P(Y_{ik} = c \mid p_{ic}) &= \phi_1 p_{ic} + (1 - \phi_1)\phi_2(c) \\ &= \Delta(p_{ic}), \end{aligned}$$

where $\Delta(p_{ic})$ is the linearly transformed category-response rate of person

i, which depends on the parameters of the forced randomized response design. Let $\mathbf{y}_{i.} = (y_{i.1}, \ldots, y_{i.C})^t$ denote the vector of observed randomized count data per response category across items for subject i. The Dirichlet-multinomial model for the observed randomized count data per category takes the form

$$\begin{aligned} \mathbf{Y}_{i.} \mid \mathbf{p}_i &\sim Mult\left(K, \Delta(\mathbf{p}_i)\right), \\ \Delta(\mathbf{p}_i) &\sim \mathcal{D}(\alpha), \end{aligned} \tag{4}$$

where $\mathbf{Y}_{i.} = (Y_{i.1}, \ldots, Y_{i.C})^t$ and $\Delta(\mathbf{p}_i) = (\Delta(p_{i1}), \ldots, \Delta(p_{iC}))^t$.

The conditional posterior distribution of the transformed category-response rate can now be stated as

$$p\left(\Delta(\mathbf{p}_i) \mid \mathbf{y}_{i.}, \alpha\right) = \frac{\Gamma(K+\alpha_0)}{\prod_c \Gamma(y_{i.c}+\alpha_c)} \prod_c (\Delta(p_{ic}))^{y_{i.c}+\alpha_c-1}.$$

Subsequently, the posterior expected (true) category-response rate can be obtained through a linear transformation. That is,

$$\begin{aligned} E\left(\Delta(p_{ic}) \mid \mathbf{y}_{i.}, \alpha\right) = \frac{y_{i.c}+\alpha_c}{K+\alpha_0} &= \Delta\left(E\left(p_{ic}\right) \mid \mathbf{y}_{i.}, \alpha\right) \\ &= \phi_1 E\left(p_{ic} \mid \mathbf{y}_{i.}, \alpha\right) + (1-\phi_1)\phi_2(c). \end{aligned} \tag{5}$$

Applying the inverse of the linear transformation on $E(\Delta(p_{ic}) \mid \mathbf{y}_{i.}, \alpha)$, the conditional posterior expected value can be obtained as

$$E(p_{ic} \mid \mathbf{y}_{i.}, \alpha) = \phi_1^{-1}\left(\frac{y_{i.c}+\alpha_c}{K+\alpha_0}\right) + (1-\phi_1^{-1})\phi_2(c).$$

The expression for the conditional posterior variance can be derived in a similar way and is equal to

$$Var\left(p_{ic} \mid \mathbf{y}_{i.}, \alpha\right) = \frac{\left(y_{i.c}+\alpha_c\right)\left(K+\alpha_0-\left(y_{i.c}+\alpha_c\right)\right)}{\phi_1^2\left(K+\alpha_0+1\right)\left(K+\alpha_0\right)^2}.$$

EMPIRICAL BAYES AND FULL BAYES ESTIMATION

There are two major approaches for estimating the model parameters when the prior parameters are unknown. An empirical Bayes approach, where the prior parameters are estimated from the marginal likelihood of the data and a full Bayes approach where hyperpriors are defined for the prior parameters and all model parameters are estimated simultaneously.

EMPIRICAL BAYES ESTIMATION

The marginal distribution of the data given the prior parameters is obtained by integrating out the category-response rates. In Appendix A, a derivation is given of the marginal likelihood of the randomized response data given the prior parameters α. This conditional distribution is given by

$$\begin{aligned} p(\mathbf{y} \mid \alpha) &= \prod_i \int_{\Delta(\mathbf{p}_i)} p(\mathbf{y}_{i.} \mid \Delta(\mathbf{p}_i))\, p(\Delta(\mathbf{p}_i) \mid \alpha)\, d(\Delta(\mathbf{p}_i)) \\ &= \prod_i \frac{K!}{\prod_c y_{i.c}!} \frac{\Gamma(\alpha_0)}{\prod_c \Gamma(\alpha_c)} \frac{\prod_c \Gamma(\alpha_c + y_{i.c})}{\Gamma(\alpha_0 + K)}. \end{aligned}$$

There are two ways of obtaining empirical Bayes estimates from this marginal likelihood. The most straightforward way is using the method of moments (Brier, 1980; Danaher, 1988; Mosimann, 1962). The second way is the method of marginal maximum likelihood (Paul, Balasooriya, and Banerjee, 2005).

Method of Moments. Let the sum of the prior parameters be α_0 and the fraction $\frac{\alpha_c}{\alpha_0}$ for each c be greater than zero. Now, the observed proportion of category responses is used to estimate the fraction $\frac{\alpha_c}{\alpha_0}$; that is,

$$N^{-1} \sum_{i=1}^{N} y_{i.c}/K = \frac{\widehat{\alpha_c}}{\alpha_0},$$

for $c = 1, \ldots, C$. The sum of the prior parameters α_0 is estimated using a relationship between the covariance matrix of the observed data, denoted as Σ_y of dimension $(C-1)(C-1)$, and of the category response rates, denoted as $\Sigma_{\Delta(p)}$ of dimension $(C-1)(C-1)$. Mosimann (1962) showed that

$$(1+\alpha_0)\Sigma_y = (K+\alpha_0)\Sigma_{\Delta(p)}. \tag{6}$$

The observed data can be used to estimate the covariance matrices; that is,

$$\widehat{\Sigma}_y = \begin{cases} (N-1)^{-1} \sum_{i=1}^{N} (y_{i.c} - \bar{y}_{..c})^2 & \text{diagonal terms,} \\ (N-1)^{-1} \sum_{i=1}^{N} (y_{i.c} - \bar{y}_{..c})(y_{i.c'} - \bar{y}_{..c'}) & \text{off-diagonal terms, } c \neq c' \end{cases}$$

and

$$\widehat{\Sigma}_{\Delta(p)} = \begin{cases} \bar{y}_{..c}\left(K - \bar{y}_{..c}\right)/K & \text{diagonal terms,} \\ -\bar{y}_{..c}\bar{y}_{..c'}/K & \text{off-diagonal terms, } c \neq c', \end{cases}$$

where $\bar{y}_{..c} = \sum_i y_{i.c}/N$. The relationship in Equation (6) can be transformed to specify a relationship between the determinants of both covariance matrices, which can be used to estimate the α_0. In this way, the estimate $\hat{\alpha}_0$ can be obtained from

$$\left(\frac{\left|\widehat{\Sigma}_y\right|}{\left|\widehat{\Sigma}_{\Delta(p)}\right|}\right)^{1/(C-1)} = \frac{K + \hat{\alpha}_0}{1 + \hat{\alpha}_0}.$$

Method of Marginal Maximum Likelihood. The Dirichlet prior parameters can also be estimated from the marginal likelihood given the observed randomized response data. The so-called marginal maximum likelihood estimates are the values for the parameters that maximize the marginal (log-)likelihood function. To facilitate the computation of marginal maximum likelihood estimates, an analytical expression is required of the marginal log-likelihood of the Dirichlet parameters given the randomized response data. The derivation of this marginal log-likelihood function is given in Appendix A. The terms not including any parameters can be ignored, which leads to the following expression

$$l(\alpha \mid \mathbf{y}) \propto \sum_{i=1}^{N} \left[\sum_{j=0}^{y_{i.1}-1} log\left(\alpha_1 + j\right) + \ldots + \sum_{j=0}^{y_{i.C}-1} log\left(\alpha_C + j\right) - \sum_{j=0}^{K-1} log\left(\alpha_0 + j\right) \right]. \tag{7}$$

The marginal maximum likelihood estimates can be obtained using the Newton-Raphson algorithm. Convergence problems of the latter are often associated with the parameter initialization step. Dishon and Weiss (1980) suggested using moment estimates as initial parameter values for the Newton-Raphson procedure.

FULL BAYES ESTIMATION

The model in Equation (4), can be extended with a hyperprior for the prior parameters. Then, the model consists of three levels, where level

1 defines the distribution of the randomized response data, level 2 the prior distribution for the level-1 parameters, and level 3 the distribution of the prior parameters. In such an hierarchical modeling approach, uncertainties are defined at different hierarchical levels. In the empirical Bayes estimation approach, the prior parameters are estimated using only the observed data, but in a full Bayes estimation approach the (hyper) prior information as well as the data are used.

In a full Bayes estimation approach all defined uncertainties can be taken into account. Therefore, a Markov chain Monte Carlo (MCMC) method will be used to estimate the posterior densities of all model parameters, which includes the transformed category response rates and the population parameters α.

To implement an MCMC procedure the collapsing property of the multinomial and Dirichlet distributions can be used. Assume that for each respondent the cells $2,\ldots,C$ are collapsed and that in total two cells are observed with $y^*_{i.2} = y_{i.2} + \ldots + y_{i.C}$. The distribution of the collapsed data are binomially distributed given the category response rate; that is,

$$p\left(y_{i.1}, y^*_{i.2} \mid \Delta(\mathbf{p}_i)\right) \propto \left(\Delta(p_{i1})\right)^{y_{i.1}} \left(1 - \Delta(p_{i1})\right)^{y^*_{i.2}}. \tag{8}$$

In the same way, the collapsing property of the Dirichlet distribution can be used. The collapsed Dirichlet prior for the transformed category response rate, $\Delta(p_{i1})$, is a beta distribution with parameters α_1 and $\alpha_0 - \alpha_1$, which leads to a beta-binomial model for the first transformed category response rate.

This procedure can also be applied to the second response category. Let $y^*_{i.3} = y_{i.3} + \ldots + y_{i.C}$ denote the collapsed data. The observed data of respondent i in category two are binomially distributed, where the responses to category one are excluded. Therefore, consider $\Delta(p_{i2})/(1 - \Delta(p_{i1}))$ as the correctly scaled success probability such that the collapsed randomized response data are binomially distributed,

$$p\left(y_{i.2}, y^*_{i.3} \mid \Delta(\mathbf{p}_i)\right) \propto \left(\frac{\Delta(p_{i2})}{1 - \Delta(p_{i1})}\right)^{y_{i.2}} \left(1 - \frac{\Delta(p_{i2})}{1 - \Delta(p_{i1})}\right)^{y^*_{i.3}}. \tag{9}$$

Subsequently, the induced beta prior has parameters α_2 and $(\alpha_0 - \alpha_1 - \alpha_2)$.

Now, the distribution of the observed data according to the multinomial distribution can be factorized as a product of binomial distributions. Let the data consist of three cells such that $K = y_{i.1} + y_{i.2} + y_{i.3}$, and let Equation

(8) and (9) define the distribution of the collapsed data sets. Then, the conditional distribution of the observed data can be given as

$$\begin{aligned}
p & \quad (\mathbf{y} \mid \Delta(\mathbf{p})) \propto \\
\propto & \quad \Delta(p_{i1})^{y_{i.1}}(1-\Delta(p_{i1}))^{K-y_{i.1}} \left(\frac{\Delta(p_{i2})}{1-\Delta(p_{i1})}\right)^{y_{i.2}} \left(1-\frac{\Delta(p_{i2})}{1-\Delta(p_{i1})}\right)^{y_{i.3}} \\
\propto & \quad \Delta(p_{i1})^{y_{i.1}}(1-\Delta(p_{i1}))^{y_{i.2}+y_{i.3}} \left(\frac{\Delta(p_{i2})}{1-\Delta(p_{i1})}\right)^{y_{i.2}} \left(\frac{\Delta(p_{i3})}{1-\Delta(p_{i1})}\right)^{y_{i.3}} \\
\propto & \quad \Delta(p_{i1})^{y_{i.1}}\Delta(p_{i2})^{y_{i.2}}\Delta(p_{i3})^{y_{i.3}},
\end{aligned}$$

which equals the unnormalized multinomial density. It can be shown in a similar way that the product of beta distributions defines the Dirichlet prior due to the collapsing property of the latter one.

This factoring of the Dirichlet-multinomial in components of beta-binomials is used in the WinBUGS (Spiegelhalter, Thomas, Best, and Lunn, 2003) implementation given in Appendix B. The implementation is given for N persons, K items, and five response categories, where the randomized response data are specified as multinomially distributed. Then, the individual category-response probabilities are specified as beta distributed, where the beta prior parameters are derived from the Dirichlet parameters.

The implementation requires the specification of a hyperprior for the Dirichlet parameters. There is often little information available about the category-response rates in the population. When a substantial number of cells does not contain observations, the parameters might not be estimable or the estimates are located on the boundary of the parameter space. A flattening prior that smooths the estimates toward a unique mode located in the interior of the parameter space is preferred when the data are sparse. The prior that assigns a common value of one or greater (say, e.g., $\alpha_c = 1$ for $c = 1, \ldots, C$) will have this smoothing or flattening property. Therefore, it might seem reasonable to restrict the prior parameters to a common value but this uninformative proper hyperprior also fixes the influence of the prior, which might be too weak for small sample sizes. It is also difficult to determine the amount of prior information given the sample information. A uniform prior, $\alpha \sim U(0,10)$, will also have this flattening property but the data will be used to estimate the prior parameters. The influence of the prior is estimated from the data. When the data are sparse, a more informative prior is needed to obtain stable parameter estimates but the data will be used to estimate the amount of prior information. Furthermore, the estimated prior parameter estimates will reveal whether the observed data do not support the model. In that case, a substantial amount of prior information is needed, more than 20% of the sample data, to obtain stable parameter estimates.

RESTRICTED DIRICHLET-MULTINOMIAL MODELING

The Dirichlet-multinomial model in Equation (4) is a saturated model in the sense that the category-response rates are freely estimated over individuals. The Dirichlet prior does not impose any restrictions that are typically present in a cross-classified data structure.

Schafer (1997) proposed a constrained Dirichlet prior to impose a loglinear model on the individual response rates. This constrained prior forms a conjugate class since it has the same functional form as the multinomial likelihood. The constrained Dirichlet prior is represented by

$$\begin{aligned}\Delta(\mathbf{p}_i) &\propto \prod_c \Delta(p_{ic})^{\alpha_c - 1} \\ \log\left(\Delta\left(\mathbf{p}_i\right)\right) &= \mathbf{M}\lambda,\end{aligned}$$

where $\mathbf{M}$ is the design matrix that defines a restriction on the transformed response rates.

In the same way, a restriction can be defined on the (true) category-response rates instead of the transformed category-response rates. It will restrict the posterior solution to that area where the loglinear model on the category response rates is true; that is, $\log\left(p_{ic}\right) = \mathbf{M}_c^t \lambda_c$, for $c = 1, \ldots, C$. Such a constrained prior makes the strong assumption that the category-response rates can be partitioned according to the implied structure. Here, such a model restriction will be particularly used to test alternative models that assume a certain homogeneity in category-response rates over individuals or groups of individuals.

APPLICATION OF THE DIRICHLET-MULTINOMIAL MODEL

A simulation study is performed to evaluate the performance of the full Bayes method for estimating the population proportions. Furthermore, the full and empirical Bayes estimates of the true individual response rates are compared under different conditions given categorical randomized response data. Then, the model is used to analyze randomized response data from the college alcohol problem scale (CAPS, O'Hare, 1997).

SIMULATION STUDY

In order to investigate the performance of the full Bayes estimation method, data were simulated under various conditions. The number of persons (N equaled 100 or 500), items (K equaled ten or fifteen), response categories (C equaled three or five), and randomizing device characteristics (ϕ_1 equaled .6 or .8) were varied. The data generation procedure comprised the following. For each respondent C category-response rates were simulated from a Dirichlet distribution given prior parameters α. The prior parameters were constant or varied over response categories. For the constant case, the sum of the prior parameters equaled C and the prior parameters equaled one such that the population proportions equaled $1/C$. For the non-constant case, the sum of the prior parameters was not equal to C and the prior parameters $(\alpha_1, \alpha_2, \alpha_3)$ equaled $(1,2,1)$ for C = 3 and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ equaled $(1,2,4,2,1)$ for $C = 5$. The simulated category-response rates were used to generate true response patterns, which were randomized using the forced response design with randomizing device probabilities ϕ_1 and $\phi_2 = 1/C$. Ten independent samples were generated for each condition.

The parameters were re-estimated using WinBUGS. The WinBUGS code of the Dirichlet-multinomial model for RR data is given in Appendix B. For each data set, 15,000 iterations were made with a burn-in period of 5,000 iterations. Each model parameter was estimated by the average of the corresponding sampled values, which is an estimate of the posterior mean.

The method was successful in model parameter estimation. In both cases, the point estimates are close to the true values and the standard deviations become smaller when increasing the number of respondents. Similar trends were found for the cases of three and five response categories. However, for the $C = 5$ case, the reduction in the estimated prior weights is better visible when increasing the number of items and/or decreasing the percentage of forced responses. This follows from the fact that more parameters need to be estimated with the same amount of observed data.

In Table 1, for $C = 5$, the estimated population proportions per category are presented. The prior parameters were divided by the sum of the prior parameters such that they were scaled in the same way as the true generating values. Note that each estimate is an average of the estimates corresponding to the ten independently generated data sets. It can be seen that the prior parameter estimates resemble the true values quite well for the constant and non-constant case. Increasing the number of persons leads to more accurate results, since the estimated standard deviations become smaller.

When decreasing the percentage of forced responses, the standard deviations remain constant for the case of ten and fifteen items, and 100 and 500 persons. The actual amount of information will increase when the amount of forced responses is reduced, since the forced responses are just random noise to mask the individual answers. From Equation (5) it can be seen that the the number of items K as well as α_0 determine the prior weight in the computation of the individual expected posterior category-response rate. It is clear that particularly for these situations the prior weights reduce since the sum of the prior parameters become smaller. That is, the influence of the population prior on the posterior mean category-response rates becomes smaller when decreasing the amount of forced responses. The observed RR data will contain more information about the individual category-response rates when less forced responses are observed and less prior information will be used to estimate the response rates. Note that the standard deviations of the sum of the prior parameters (α_0) become smaller when decreasing the number of forced responses. The typical advantage of the full Bayes estimation method applies here, where the prior weights are also estimated from the data. Note that for $\phi_1 = .6$, 40% of the data are forced responses. So, the actual amount of information in the data is rather limited but the population parameters can still be recovered. The decrease in the amount of forced responses leads to more accurate results.

When increasing the number of items, the standard deviations only become slightly smaller. The additional amount of five RR observations did not led to a substantial increase in the precision of the posterior mean estimates. The increase in items led in to a higher estimate of α_0 with a smaller standard deviation. As a result, the posterior mean response rates will be more influenced by the data than the prior information due to the higher number of items and the higher posterior mean estimate of α_0.

Table 1: Full Bayes estimates of population proportions for five response categories, 100 and 500 respondents, and 10 and 15 items.

		K = 10				K = 15			
		Const.		Non-Const.		Const.		Non-Const.	
	Parameter	Mean	SD	Mean	SD	Mean	SD	Mean	SD
N = 100									
$\phi_1 = 0.6$									
	α_1/α_0	.209	.016	.144	.013	.207	.014	.145	.012
	α_2/α_0	.203	.016	.200	.015	.205	.014	.205	.013
	α_3/α_0	.197	.016	.318	.018	.197	.014	.308	.015
	α_4/α_0	.195	.016	.198	.015	.195	.014	.195	.013
	α_5/α_0	.196	.016	.140	.013	.195	.014	.147	.012
	α_0	12.003	1.642	15.615	2.225	12.082	1.432	17.445	2.204
$\phi_1 = 0.8$									
	α_1/α_0	.199	.017	.123	.013	.206	.016	.114	.011
	α_2/α_0	.207	.017	.208	.016	.194	.015	.201	.014
	α_3/α_0	.190	.017	.350	.019	.202	.016	.357	.017
	α_4/α_0	.205	.017	.204	.016	.201	.016	.197	.014
	α_5/α_0	.200	.017	.116	.012	.197	.016	.119	.011
	α_0	7.785	1.008	12.217	1.679	7.984	.861	14.183	1.781
N = 500									
$\phi_1 = 0.6$									
	α_1/α_0	.196	.007	.140	.006	.204	.006	.141	.005
	α_2/α_0	.197	.007	.203	.007	.201	.006	.201	.006
	α_3/α_0	.202	.007	.321	.008	.201	.006	.319	.007
	α_4/α_0	.201	.007	.195	.007	.196	.006	.200	.006
	α_5/α_0	.203	.007	.141	.006	.198	.006	.139	.005
	α_0	14.276	1.116	22.994	2.083	15.534	1.001	26.183	2.057
$\phi_1 = 0.8$									
	α_1/α_0	.197	.008	.122	.006	.200	.007	.122	.005
	α_2/α_0	.202	.008	.201	.007	.202	.007	.198	.006
	α_3/α_0	.203	.008	.357	.008	.200	.007	.357	.007
	α_4/α_0	.198	.008	.201	.007	.200	.007	.201	.006
	α_5/α_0	.200	.008	.119	.005	.198	.007	.122	.005
	α_0	8.351	.526	15.232	1.265	8.672	.454	16.511	1.110

In this simulation study, the posterior mean response rates were also compared for different number of respondents (100 and 500) and different prior values. To evaluate the prior influence, individual category-response rates were estimated using the simulated and estimated prior parameters, and through a full Bayes estimation method. Per response category, the mean squared error (MSE) of the estimated response rates was calculated. The MSE comprises a comparison of the estimate of the individual response rate with the true value. For category c, the MSE can be stated as

$$MSE(\hat{\mathbf{p}}_c \mid \mathbf{y}) = \sum_{i=1|c}^{N} E(p_{ic} - \hat{p}_{ic})^2 + \sum_{i=1|c}^{N} Var(\hat{p}_{ic})^2,$$

where the first term is the cumulative bias between the true value and its estimate and the second term represents the cumulative variance of the estimate.

In Table 2, the MSEs are presented for five response categories, where the cumulative bias and variance terms are given in parenthesis.

Table 2: Estimated MSEs concerning response-rate estimates for five response categories, and 100 and 500 respondents. The bias and variance components of the MSE are given in parenthesis, respectively.

	Category	$MSE(\hat{p}_{EB}, \alpha_c)$	$MSE(\hat{p}_{EB}, \alpha_c = 1)$	$MSE(\hat{p}_{FB})$
N=100				
	c = 1	.45 (.44, .00)	.52 (.51, .01)	.90 (.42, .48)
	c = 2	.66 (.66, .01)	.85 (.84, .01)	1.33 (.59, .74)
	c = 3	1.18 (1.16, .02)	1.53 (1.50, .02)	2.11 (1.01, 1.09)
	c = 4	.90 (.89, .01)	1.05 (1.03, .02)	1.67 (.88, .79)
	c = 5	.51 (.51, .00)	.56 (.55, .01)	.98 (.47, .50)
N = 500				
	c = 1	2.22 (2.20, .02)	3.12 (3.08, .04)	4.27 (1.95, 2.32)
	c = 2	4.29 (4.24, .05)	5.32 (5.25, .07)	7.66 (4.00, 3.66)
	c = 3	6.24 (6.15, .09)	7.91 (7.79, .12)	10.78 (5.62, 5.16)
	c = 4	4.34 (4.30, .05)	5.22 (5.15, .07)	7.63 (4.14, 3.49)
	c = 5	2.51 (2.49, .02)	3.18 (3.14, .40)	4.54 (2.26, 2.27)

Two empirical Bayes estimates are considered, the $MSE(\hat{p}_{EB}, \alpha_c = 1)$ and the $MSE(\hat{p}, \alpha_c)$, which is based on prior parameters that are set to one, denoted as the homogenous prior, and the simulated prior parameters, respectively. The $MSE(\hat{p}_{FB})$ is based on full Bayes estimates using the WinBUGS program. The simulated prior parameters correspond to the non-constant case with $\alpha = (1, 2, 4, 2, 1)$, which differs from the homogenous prior with prior parameters equal to one.

For 100 and 500 persons, the bias is smallest for the full Bayes estimates and they are slightly better than empirical Bayes estimates given the true prior parameters. The full Bayes estimates are accurate with respect to bias since the prior weights are also estimated from the sampled data. The estimates using the homogenous prior, with all prior parameters equal to one, have the highest bias but smaller MSEs than those based on the full Bayes estimates. For the latter one, the estimated variances are much higher due to the fact that the prior parameters also need to be estimated. Therefore, the homogenous prior leads to quite accurately estimated category-response rates and performs better than the full Bayes estimates given the MSEs.

RESPONSE RATES OF ALCOHOL-RELATED NEGATIVE CONSEQUENCES

The college alcohol problem scale (CAPS; OHare, 1997) was used to measure frequencies of alcohol-related negative consequences among college students. Thirteen items of the CAPS scale were used that covered socio-emotional problems (hangovers, memory loss, nervousness, depression) and community problems (drove under the influence, engaged in activities related to illegal drugs, problems with the law). Each item has a five-point scale (one = never/almost never, five =almost always).

A total of 793 US college student were at random divided in two groups. One group of 351 participants answered the questionnaire directly without using a randomizing device, denoted as the direct-questioning (DQ) group. The other group, denoted as the RR group, consisted of 442 participants and they responded to the questionnaire according to a forced randomized response design, where $\phi_1 = .60$ and $\phi_2(c) = .20$ for $c = 1, \ldots, 5$. The RR group used a spinner to answer the questions. The spinner was developed such that 60% of the area was comprised of answer honestly space, and 40% of the area was divided into equal sections to represent the five possible answer choices.

The main focus of the study was to investigate whether the RR technique improved the accuracy of self-reports. The sensitivity of the response categories was evaluated, where it was expected that a strong confirmation to an item is more sensitive than a negative confirmation. The Dirichlet-multinomial model with a restricted Dirichlet prior was used to evaluate the effect of the RR technique per category, where the between-group differences in mean category-response rates were investigated.

Group-Specific Population Proportions. The Dirichlet-multinomial model was estimated using group-specific population proportions such that the DQ and the RR group each had their own prior parameters. In Table 3, the full Bayes mean estimates of the population proportions and standard deviations are presented. Note that the estimated population proportions of the RR group are transformed such that they estimate the true category proportions in this group. The estimated population proportions corresponding to the observed response rates are given in parenthesis, which also take the forced responses into account. The standard deviations are given in the next column, where the standard deviations of the non-transformed proportions are given in parenthesis.

Table 3: CAPS: Estimated population proportions per category for the DQ and RR group.

	DQ (351)		RR (442)	
	Mean	SD	Mean	SD
α_1/α_0	.590	.010	.471(.363)	.012(.007)
α_2/α_0	.161	.007	.170(.182)	.010(.006)
α_3/α_0	.129	.006	.153(.172)	.009(.006)
α_4/α_0	.069	.005	.131(.159)	.009(.005)
α_5/α_0	.051	.003	.075(.125)	.008(.005)
α_0	9.872	.648	20.360	1.672

It can be seen that the estimated population proportion of category one of the DQ group is higher and that the other category-mean proportions are smaller compared to the RR group. This indicates that the respondents in the DQ group are less likely to confirm experiences with alcohol-related negative consequences. However, the respondents were randomly assigned to the RR group, which suggests serious underreporting in the DQ group. Although the RR group contains more respondents than the DQ group, the standard deviations of the estimated population proportions of the RR group are higher, since 40% of the data are forced responses. The standard deviations of the non-transformed population proportions are comparable since they are based on all data. The RR group has a higher estimate of the prior weight (α_0) compared to the DQ group due to the number of forced responses. As a result, the posterior means of the individual category-response rates are more influenced by the prior in the RR group than in the DQ group.

Linear Restricted Category Response Rates. To investigate the category-specific effect of the RR method, a loglinear model was defined for the category-response rates. For each category, the logarithm of the true category-response rates were explained by a constant and a category-specific RR effect. This restriction of the Dirichlet prior is given by

$$log(p_{ic}) = \lambda_{0c} + \lambda_{1c} RR_i,$$

for $c = 1, \ldots, C$, where RR_i equals one when respondent i belongs to the RR group and zero otherwise. Note that the loglinear representation was only used to evaluate the category-specific RR effect.

In Table 4, the intercept and RR effect are presented per category for the DQ and RR group. The 95% highest posterior density (HPD) region is also given for each parameter. The estimates can be used to compute the posterior expected population proportion per category in each group. It follows that about $\exp(-.62) = 54\%$ and $\exp(-.62 - .27) = 41\%$ in the DQ group and the RR group, respectively, say that they almost never experience negative consequences of drinking. The other posterior mean percentages can be computed in a similar way.

Table 4: CAPS: Category-specific intercepts and RR effects.

	Intercept			RR Effect		
	Mean	SD	HPD	Mean	SD	HPD
1 Almost never	-.62	.03	-.68, -.56	-.27	.03	-.33, -.22
2 Seldom	-2.13	.07	-2.28,-2.00	.43	.08	.28, .57
3 Sometimes	-2.39	.08	-2.56,-2.23	.65	.08	.49, .81
4 Often	-3.61	.16	-3.93,-3.33	1.65	.13	1.39, 1.91
5 Almost always	-4.71	.18	-5.09,-4.37	2.70	.13	2.45, 2.94

The RR effect is negative for the first category (labeled almost never) and positive for all other effects. From the HPD regions follow that all RR effects are significantly different from zero. It can be concluded that in the DQ group, respondents underreported any experiences of negative consequences and, subsequently, overreported that almost never negative consequences were experienced. Furthermore, the estimated RR effects increase with an increase in the number of negative experiences, where the fifth category has the highest RR effect. Thus, an increase in the number of experiences of alcohol-related negative consequences leads to a more sensitive response option. The difference between the groups with respect to the posterior expected proportion of respondents that admits experiencing negative consequences is highest for the fifth response option. In that case, around 1% of the DQ group admits to have almost always alcohol-related negative consequences, which is around 13% in the RR group. It can be concluded that the RR technique led to a higher degree of cooperation and more accurate data, especially when the response options become more sensitive.

The response data from the RR group were used to explore ethnic differences in experiencing alcohol-related negative consequences. The responses from the DQ group were shown to be biased, since effects of under- and overreporting were found. In this study, the racial origin of the respondents was administered. The RR group consisted of 2% Asians, 83% white Americans, 11% African Americans, and 12% belonged to another race. An indicator variable, labeled Ethnicity, was used in the log-linear model that represented the racial origin of each respondent.

In Table 5, the estimated category-specific effects of ethnicity on the individual category response rates are given. Each posterior mean effect is accompanied with a posterior standard deviation (SD) and a 95% highest posterior density (HPD) interval. Under the column labeled Effect, the estimated effects are represented, where each category-specific intercept represents the average population level on the logarithmic scale. Under the column labeled Scaled Effect, the intercept represents the average population level of the African Americans on the logarithmic scale. The scaled ethnic effect of the this group is in that case equal to zero.

For the first category, the African Americans score significantly higher than the other groups. Furthermore, the estimated scaled effects of the other groups are significantly smaller. This means that the percentage of African-Americans experiencing almost never negative consequences is much higher compared to the other groups. In the same way, it follows that the percentage of African-Americans experiencing negative consequences seldom to almost always is lower than that of other groups. In the second category, the white Americans also score significantly higher than the African Americans, which follows from the scaled effects. The third to

Table 5: CAPS: Ethnic differences per response category.

		Effect			Scaled Effect		
Cat.	Ethnicity	Mean	SD	HPD	Mean	SD	HPD
1							
	Asian	-.04	.12	-.29, .17	-.18	.16	-.51, .11
	White American	-.03	.05	-.12, .07	-.16	.05	-.26,-.06
	African American	.13	.06	.02, .24	.00		
	Other	-.06	.08	-.22, .08	-.20	.10	-.40,-.02
2							
	Asian	.25	.19	-.14, .60	.46	.26	-.08,1.00
	White American	-.02	.08	-.17, .13	.19	.09	.00, .37
	African American	-.21	.10	-.40,-.03	.00		
	Other	-.01	.12	-.26, .22	.20	.16	-.14, .50
3							
	Asian	-.23	.20	-.60, .17	-.28	.27	-.80, .25
	White American	.07	.08	-.09, .22	.03	.09	-.15, .20
	African American	.04	.10	-.15, .23	.00		
	Other	.11	.12	-.12, .36	.08	.16	-.23, .39
4							
	Asian	.13	.20	-.27, .50	.22	.27	-.30, .76
	White American	.06	.08	-.10, .21	.15	.09	-.02, .32
	African American	-.09	.10	-.29, .09	.00		
	Other	-.10	.12	-.34, .13	-.01	.16	-.33, .28
5							
	Asian	-.01	.19	-.37, .35	.07	.25	-.43, .55
	White American	-.03	.07	-.18, .10	.04	.08	-.12, .19
	African American	-.08	.09	-.25, .09	.00		
	Other	.12	.11	-.11, .33	.19	.14	-.09, .46

fifth response categories did not show any significant ethnic differences.

DISCUSSION

For small data samples, the Dirichlet-multinomial model is proposed for categorical RR data, where a linear transformation of the response rates is implemented to adjust for the RR sampling design. This model is a generalization of the beta-binomial model for binary RR data. Both models are suitable for sensitive-survey studies and small data samples. The individual category-response rates are related to the observed data, but a linear transformation can be used to derive the true categorical-response rates. The parameters of this linear transformation are the characteristics of the randomizing device and they are usually known. The derived expressions of the posterior expectation and variance of the category-response rates are useful in case of empirical Bayes estimation or explicit prior knowledge about response rate population parameters.

The idea of full Bayes parameter estimation was elaborated using the synthetic data set. The simulation study has shown that full Bayes model was able to rather accurately estimate values of parameters for different number of response categories. The method was equally successful in retrieving the parameters for the constant case of homogenous prior parameters as well as for the case of non-constant prior parameters. Moreover, the simulation study concluded that increasing the number of persons leads to more accurate results, while the variation of the percentage of forced responses does not influence the accuracy.

A constrained-Dirichlet prior is used to identify homogeneity in response rates over items and persons. Therefore, the WinBUGS program was extended to define a constrained-Dirichlet prior, where a loglinear model was defined on the true category-response rates.

An important effect was identified in the real data study, which showed that the effect of the RR method varied over response categories. A priori it was assumed that the response options varied in their sensitivity, where a higher degree of accordance with the sensitive item but this hypothesis was never tested in the literature. The analysis showed a substantial increase in agreement with more sensitive response options under the randomized response condition.

In RR studies the topic of compliance is often an issue. Respondents are instructed to follow the RR instructions but may for different reasons act differently. In large-scale sample studies, a latent class structure can be integrated in the model to identify non-compliant behavior. The responses from the non-compliant subjects are modeled differently. The Dirichlet-multinomial model can also be extended with a two-component latent-class structure to allow for non-compliance, but that would require more response data to obtain stable parameter estimates.

REFERENCES

Brier, S. S. (1980). Analysis of contingency tables under cluster sampling. *Biometrika, 67*, 591−596.

Böckenholt, U., & van der Heijden, P. G. M. (2007). Item-randomized response models for measuring noncompliance: Risk-return perception, social influences, and self-protective responses. *Psychometrika, 72*, 245−262.

Danaher, P. J. (1988). Parameter estimation for the Dirichlet-multinomial distribution using supplementary beta-binomial data. *Communications in Statistics - Theory and Methods, 17*, 1777−1788.

Dishon, M., & Weiss, G. H. (1980). Small sample comparison of estimation methods for the Beta distribution. *Journal of Statistical Computation and Simulation, 11*, 1−11.

De Jong, M. G., Pieters, R., & Fox, J.-P. (2010). Reducing social desirability bias through item randomized response: An application to measure underreported desires. *Journal of Marketing Research, 47*, 14−27.

Edgell, S. E., Himmelfarb, S., & Duchan, K. L. (1982). The validity of forced responses in a randomized response model. *Sociological Methods and Research, 11*, 89−100.

Fox, J.-P. (2005). Randomized item response theory models. *Journal of Educational and Behavioral Statistics, 30*, 189−212.

Fox, J.-P. (2008). Beta-binomial ANOVA for multivariate randomized response data. *British Journal of Mathematical and Statistical Psychology, 61*, 453−470.

Fox, J.-P., & Wyrick, C. (2008). A mixed effects randomized item response model. *Journal of Educational and Behavioral Statistics, 33*, 389−415.

Greenberg, B. G., Abul-Ela, A.-L. A., Simmons, W. R., & Horwitz, D. G. (1969). The unrelated question randomized response model: Theoretical framework. *Journal of the American Statistical Association, 64*, 520−539.

Goodhardt, G. J., Ehrenberg, A. S. C., & Chatfield, C. (1984). The Dirichlet: A comprehensive model of buying behavior. *Journal of the Royal Statistical Society. Series A (General), 147*, 621−655.

Mosimann, J. E. (1962). On the compound multinomial distribution, the multivariate β-distribution, and correlations among proportions. *Biometrika, 49*, 65−82.

O'Hare, T. M. (1997). Measuring problem drinkers in first time offenders: Development and validation of the Collage Alcohol Problem Scale (CAPS). *Journal of Substance Abuse Treatment, 14*, 383−387.

Paul, S. R., Balasooriya, U., & Banerjee, T. (2005). Fisher information matrix of the Dirichlet-multinomial distribution. *Biometrical Journal, 47*, 230−236.

Schafer, J. L. (1997). *Analysis of incomplete multivariate data.* London: Chapman & Hall.

Spiegelhalter, D. J., Thomas, A., Best, N., & Lunn, D. (2003). *WinBUGS User Manual, Version 1.4* Retrieved from http://www.mrc-bsu.cam.ac.uk /bugs/winbugs/manual14.pdf.

Tourangeau, R., Rips, L. J., & Rasinski, K. (2000). *The psychology of survey response.* Cambridge, England: Cambridge University Press.

Tourangeau, R., & Yan, T. (2007). Sensitive questions in surveys. *Psychological Bulletin, 133*, 859−883.

Warner, S. L. (1965). Randomized response: A survey technique for eliminating evasive answer bias. *Journal of the American Statistical Association, 60*, 63−69.

Wilson, J. R., & Chen, G. S. C. (2007). Dirichlet-multinomial model with varying response rates over time. *Journal of Data Science, 5*, 413−423.

A Derivation of the Marginal Log-Likelihood Function

Here, the derivation of the marginal likelihood expressed by Equation (7) is given. The marginal distribution of the RR data given the prior parameters α can be stated as

$$
\begin{aligned}
p(\mathbf{y} \mid \alpha) &= \prod_i \int_{\Delta(\mathbf{p}_i)} p\left(\mathbf{y}_{i.} \mid \Delta(\mathbf{p}_i)\right) p\left(\Delta(\mathbf{p}_i) \mid \alpha\right) d(\Delta(\mathbf{p}_i)) \\
&= \prod_i \frac{K!}{\prod_c y_{i.c}!} \frac{\Gamma(\alpha_0)}{\prod_c \Gamma(\alpha_c)} \int_{\Delta(\mathbf{p}_i)} \prod_c \Delta(p_{ic})^{\alpha_c + y_{i.c} - 1} d(\Delta(\mathbf{p}_i)) \\
&= \prod_i \frac{K!}{\prod_c y_{i.c}!} \frac{\Gamma(\alpha_0)}{\prod_c \Gamma(\alpha_c)} \frac{\prod_c \Gamma(\alpha_c + y_{i.c})}{\Gamma(\alpha_0 + K)}.
\end{aligned}
$$

The gamma function Γ can be represented as a factorial function, where $\Gamma(n) = (n-1)!$. Therefore, the marginal distribution can be rewritten in terms of factorial multipliers.

$$
p(\mathbf{y} \mid \alpha) = \prod_i \frac{K!}{\prod_c y_{i.c}!} \frac{(\alpha_0 - 1)!}{\prod_c (\alpha_c - 1)!} \frac{\prod_c (\alpha_c + y_{i.c} - 1)!}{(\alpha_0 + K - 1)!}.
$$

The factorial multipliers of the last fraction can be manipulated such that

$$
\begin{aligned}
(\alpha_c + y_{i.c} - 1)! &= \left[\prod_{j=1}^{y_{i.c}} ((\alpha_c - 1) + j) \right] (\alpha_c - 1)! \\
&= \left[\prod_{j=0}^{y_{i.c}-1} (\alpha_c + j) \right] (\alpha_c - 1)!
\end{aligned}
$$

and

$$
\begin{aligned}
(\alpha_0 + K - 1)! &= \left[\prod_{j=1}^{K} ((\alpha_0 - 1) + j) \right] (\alpha_0 - 1)! \\
&= \left[\prod_{j=0}^{K-1} (\alpha_0 + j) \right] (\alpha_0 - 1)!.
\end{aligned}
$$

The density $p(\mathbf{y} \mid \alpha)$ can be rewritten as

$$
p(\mathbf{y} \mid \alpha) = \prod_i \frac{K!}{\prod_c y_{i.c}!} \frac{\left[\prod_{j=0}^{y_{i.1}-1} (\alpha_1 + j) \right] \cdots \left[\prod_{j=0}^{y_{i.C}-1} (\alpha_C + j) \right]}{\prod_{j=0}^{K-1} (\alpha_0 + j)}
$$

and the the logarithm of the density $p(\mathbf{y} \mid \alpha)$ can be stated as

$$
l(\alpha \mid \mathbf{y}) \propto \sum_{i=1}^{N} \left[\sum_{j=0}^{y_{i.1}-1} log\left(\alpha_1 + j \right) + \ldots + \sum_{j=0}^{y_{i.C}-1} log\left(\alpha_C + j \right) - \sum_{j=0}^{K-1} log\left(\alpha_0 + j \right) \right],
$$

leaving out the first term, which is a constant.

B WinBUGS Code of the Multinomial-Dirichlet Model

The code of the Dirichlet-multinomial model for RR data is given for N persons, K items, and five response categories. The randomizing device has parameters ϕ_1 and ϕ_2.

```
model
{
     for (i in 1:N)
     {
          y[i,1:5]~dmulti(q[i,],K)
          q[i,1]~dbeta(alpha[1],betatot1)
          q2_star[i]~dbeta(alpha[2],betatot2)
          q[i,2]<-q2_star[i]*(1-q[i,1])
          q3_star[i]~dbeta(alpha[3],betatot3)
          q[i,3]<-q3_star[i]*(1-q[i,1]-q[i,2])
          q4_star[i]~dbeta(alpha[4],alpha[5])
          q[i,4]<-q4_star[i]*(1-q[i,1]-q[i,2]-q[i,3])
          q[i,5]<-1-q[i,1]-q[i,2]-q[i,3]-q[i,4]
     }

     alpha[1]~dunif(0,10)
     alpha[2]~dunif(0,10)
     alpha[3]~dunif(0,10)
     alpha[4]~dunif(0,10)
     alpha[5]~dunif(0.5,10)

     alpha0<-sum(alpha[1:5])

     betatot1<-sum(alpha[2:5])
     betatot2<-sum(alpha[3:5])
     betatot3<-sum(alpha[4:5])

     for (i in 1:N)
     { for (c in 1:5)
          {
               p[i,c] <- (q[i,c] - (1-phi1)*phi2) / phi1
          }
     }
}
```